0J2 - Mechanics Lecture Notes 5

0J2 Introduction to Statics

Statics is the study of forces in situations where no motion occurs. Since there is no motion in statics any forces must balance out. This situation is called equilibrium.

Force. As we saw in dynamics a force is something which acts upon a body so as to change (in dynamics) or try to change (in statics) its speed or direction of motion. In other words to give it an acceleration. In dynamics the force and the acceleration are related by Newton's second law $\mathbf{F} = m\mathbf{a}$.

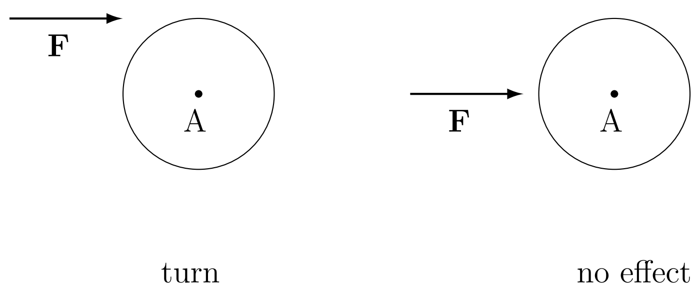
Force is measured in newtons (N), one newton being the force required to produce an acceleration of 1ms^{-2} for an object with mass 1kg.

Since force is applied in a certain direction it is a vector quantity. To specify a force we need to give:-
(i) magnitude (*i.e.* how many newtons)
(ii) direction.
(iii) the point of application of the force.

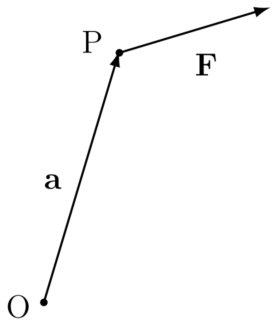
The first two of these are a consequence of it being a vector.

(ii) and (iii) determine the 'line of action' of the force. The line of action is very important in practice.

e.g. Consider a wheel with fixed centre A. The same force **F** is applied at two different points as shown. In one case the wheel turns. In the other it does not.



If a force $\mathbf{F}$ is applied at point P with position vector $\mathbf{a}$ then the equation of the line of action of the force is



$$\mathbf{r} = \mathbf{a} + t\mathbf{F}$$

where t is the parameter, since the direction of the line of action is the same as the direction of $\mathbf{F}$.

In the study of mechanics we use a number of idealised objects:-

1) Particles: These have a mass but negligible size so they exist at a point.

2) Rigid body: A body with a size which is not deformed when a force is applied.

3) Laminar body: A 2-dimensional body, *i.e.* thickness zero.

4) Light inextensible string: A string with zero mass which does not stretch when pulled.

5) Smooth surface: One with no friction.

For equilibrium we need the forces to balance in every direction. Since any 2D vector can be written as the sum of two perpendicular vectors it is sufficient for the forces to balance in any two perpendicular directions. In 3D we need them to balance in three mutually perpendicular directions.

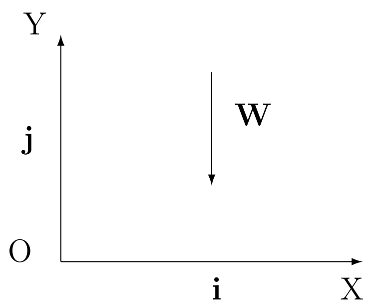
Types of forces

For this course we consider three types of force.

1) Gravity: This is the force between two bodies due to their mass.
When one body is the earth then the force on the other body is called its weight W.

We assume that weight always acts vertically downwards.
As we saw earlier the force of gravity acting on a mass of m kg is mg N so $W = mg$, where $g = 9.81$ ms^{-2}.

Notation Often we work in 2D in which case the Y-axis is usually chosen to be vertically upwards so the the weight acts in the negative $\mathbf{j}$ direction.



$$\mathbf{W} = -W\mathbf{j} = -mg\mathbf{j} = (0, -W)$$

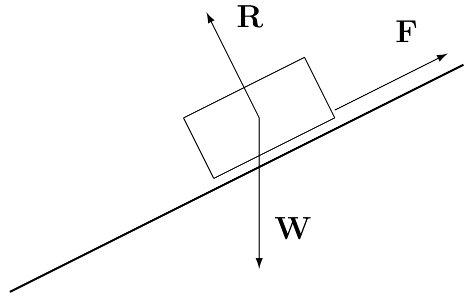
In 3D we usually take the Z-axis vertically upwards so

$$\mathbf{W} = -W\mathbf{k} = -mg\mathbf{k} = (0, 0, -W)$$

2) Contact forces (*i.e.* Push): Caused by one body pushing on another. A special case of this is normal reaction.
If a book rests upon a table then the weight acts downwards. The book does not move because the table pushes upwards by an equal amount. This force is always perpendicular to the surface so it is called the normal reaction.

Another special case is friction which hinders motion along a surface and therefore acts parallel to the surface.

e.g. A car parked on a hill.



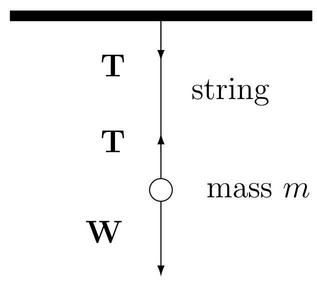
W is the weight.
R is the normal reaction.
F is the friction.

The car does not move because these three forces balance out.

(Sometimes the vector sum of **R** and **F** is called the total reaction.)

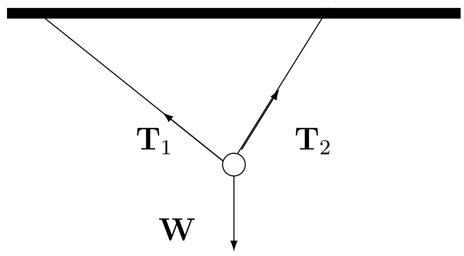
3) Forces of attachment (*i.e.* pull): Typically these are forces caused by tension in a string attached to a body. They always act along the string.

e.g. 1. A particle of mass m suspended by a string from a ceiling.



Weight **W** acts downwards. Tension in the string is a force **T** upwards acting on the mass m and a force **T** downwards acting on the ceiling.

e.g. 2. A particle suspended by a two strings of unequal length.



Two strings with tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ support a body with weight **W**.

A smooth pulley can be used to change the orientation of the string and hence the direction of the tension.

Force Diagrams Usually it is much easier to see what is happening by drawing a diagram showing the forces.

— This was done for *e.g.* 1 and *e.g.* 2 above and also for the car on the hill.

— Make sure the diagram is large enough to see the different forces clearly!

Resolving Forces

As a force is a vector it has components in different directions. Often we use the standard Cartesian form:-

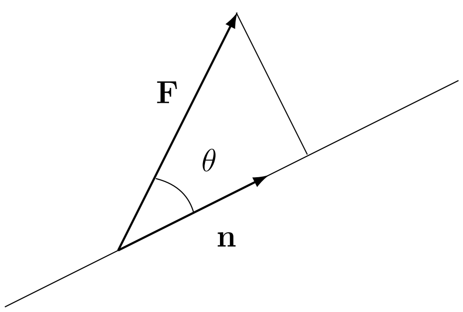
$$\begin{aligned} \mathbf{F} &= F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \qquad \text{in 3D} \\ \mathbf{F} &= F_1\mathbf{i} + F_2\mathbf{j} \qquad \text{in 2D} \end{aligned}$$

F_1 is the component in the X-direction *etc.*

Note that $F_1 = \mathbf{F}.\mathbf{i}$ so that the component in the X-direction is obtained by taking the scalar product of $\mathbf{F}$ with the unit vector in the X-direction $\mathbf{i}$.

Similarly $F_2 = \mathbf{F}.\mathbf{j}$ and $F_3 = \mathbf{F}.\mathbf{k}$.

We can get the component in any other direction by taking the scalar product of $\mathbf{F}$ with a unit vector in that direction, *i.e.* if $\mathbf{n}$ is a unit vector in any direction then $\mathbf{F}.\mathbf{n}$ gives the component of $\mathbf{F}$ in that direction.



θ is the angle between $\mathbf{F}$ and $\mathbf{n}$.

$$\mathbf{F}.\mathbf{n} = F \times 1 \times \cos\theta$$

Component of $\mathbf{F}$ in the direction of $\mathbf{n}$ is $F\cos\theta = \mathbf{F}.\mathbf{n}$.

Example 1. Find the component of $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j}$ in the direction of $\mathbf{a} = \mathbf{i} + \mathbf{j}$.

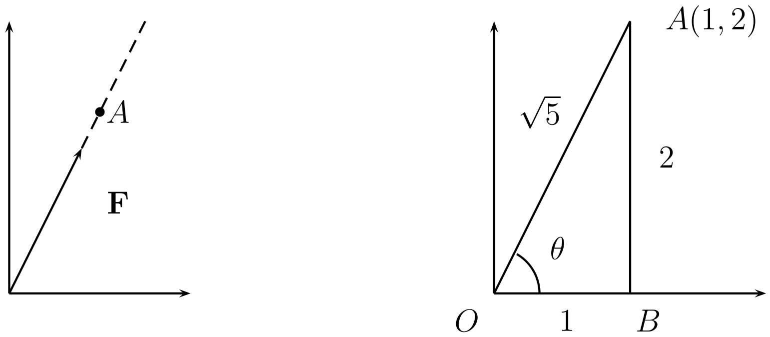
$\mathbf{a}$ is not a unit vector. Magnitude $a = |\mathbf{a}| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

A unit vector in the direction of $\mathbf{a}$ is

$$\mathbf{n} = \frac{1}{a}\mathbf{a} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}).$$

Hence component is $\quad \mathbf{F}.\mathbf{n} = 2\frac{1}{\sqrt{2}} + 3\frac{1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$ newtons.

Example 2. Force $\mathbf{F}$ has magnitude 10 N. It acts along the line $y = 2x$. Find the component in the X-direction.



Let A be the point $(1, 2)$ on the line. The distance $OA = \sqrt{5}$.
Let the angle between the line and the X-axis be θ.

Clearly $\cos\theta = \dfrac{1}{\sqrt{5}}$.

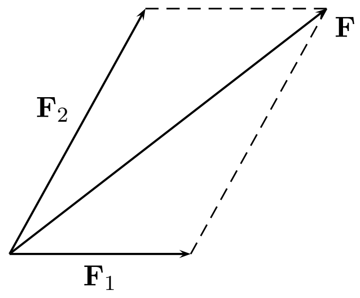
The component in the X-direction is $F\cos\theta = \dfrac{10}{\sqrt{5}} = 4.47$ N.

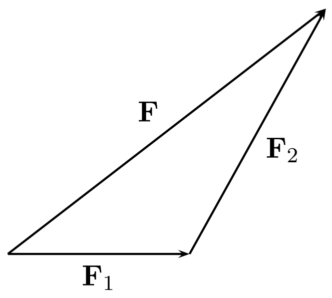
Addition of Forces

If two or more forces act on a particle, the point of application is the same for all of them, so the line of action, property (iii), is not so important in this case. The vector sum of all the forces acting on the particle is called the resultant.

If two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on a particle, the total effect is the same as force $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$.

This is a vector sum, so we use the parallelogram or triangle rules.





Similarly, for 3 or more forces:

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$$

In practice, we often use components to find the sum (rather than drawing diagrams).

Example   Find the resultant of

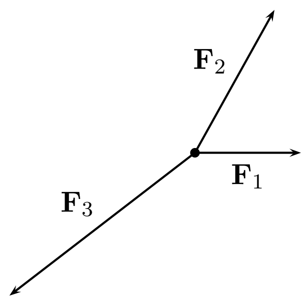
$$\begin{aligned} \mathbf{F}_1 &= 3\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k}\ \text{N} &= (3, -2, 4) \\ \mathbf{F}_2 &= -\,\mathbf{i} + 3\,\mathbf{k}\ \text{N} &= (-1, 0, 3) \end{aligned}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = (2, -2, 7) = 2\,\mathbf{i} - 2\,\mathbf{j} + 7\,\mathbf{k}\ \text{N}$$

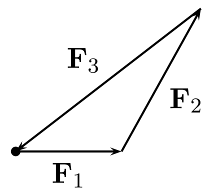
Equilibrium of Particles

If the total force $\mathbf{F}$ is zero then there is no overall force on the particle, and it is said to be in *equilibrium*.

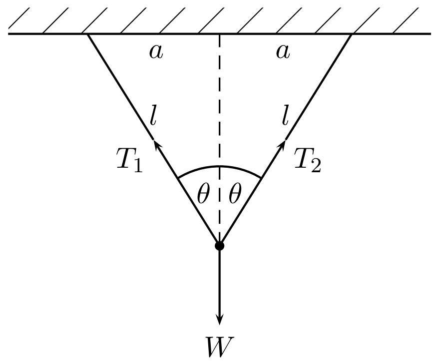
Example: 3 forces



If we add these using the 'triangle' rule the total force is $\mathbf{0}$. So the force polygon (here a triangle since there are three forces) is **closed**:



Example 1: Two strings of equal length l support a weight W as shown.



The points of attachment are $2a$ apart. Find the tension in the strings.

Answer: From the diagram we see $\sin\theta = a/l$.

The total force on the mass is zero. We resolve in vertical and horizontal directions. (Clearly if the force is zero its component in any direction is zero.)

Horizontally: $T_2 \sin\theta - T_1 \sin\theta = 0$; hence $T_2 = T_1$.

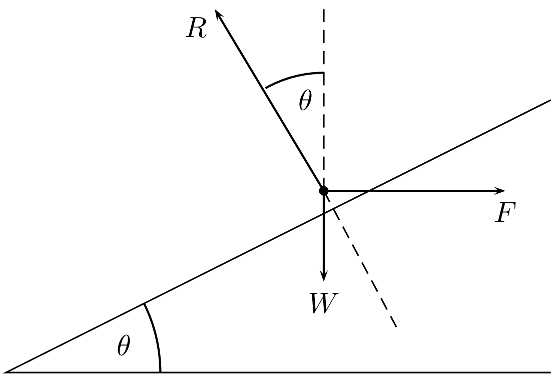
(Clearly this is correct because of symmetry.)

Vertically: $T_2 \cos\theta + T_1 \cos\theta - W = 0$; which gives $2T_1 \cos\theta - W = 0$, and so $T_1 = W/(2\cos\theta)$.

$$\text{Now} \quad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{a^2}{l^2}} = \frac{\sqrt{l^2-a^2}}{l} \qquad \text{so} \qquad T_1 = \frac{lW}{2\sqrt{l^2-a^2}}.$$

Usually we solve these problems by resolving in two perpendicular directions (though not always vertical and horizontal, as here).

Example 2: A particle of weight W rests on a *smooth* (i.e. no friction) plane inclined at an angle θ to the horizontal. Find the force $\mathbf{F}$ *applied horizontally* to produce equilibrium.



Three forces act on the particle: weight $\mathbf{W}$, normal reaction $\mathbf{R}$ and applied force $\mathbf{F}$.

Method A: Resolve horizontally and vertically.

Vertically: $R\cos\theta - W = 0$ hence $R = W/\cos\theta$

Horizontally: $F - R\sin\theta = 0$ which means

$$F = R\sin\theta = W\frac{\sin\theta}{\cos\theta} = W\tan\theta.$$

Method B: Resolve parallel and perpendicular to plane.

Parallel to plane: $F\cos\theta - W\sin\theta = 0$ hence $F = W\tan\theta$ (as before).

Perpendicular to plane: $R - F\sin\theta = W\cos\theta$ (which we could use to find R; we do not actually need this here).

Friction in Statics

Friction is very important is statics—it is often responsible for producing equilibrium where otherwise there would be motion.

Friction acts to oppose the motion of a body across a surface with which it is in contact (i.e. sliding motion). In experiments it is found to follow (approximately) certain rules:

(i) Friction acts parallel to the surface, and its direction opposes motion.

(ii) The magnitude of the friction is *just enough* to prevent motion. If the tendency to move increases (e.g. because other forces increase), the friction increases to balance this.

But the friction can only increase up to a limiting value.

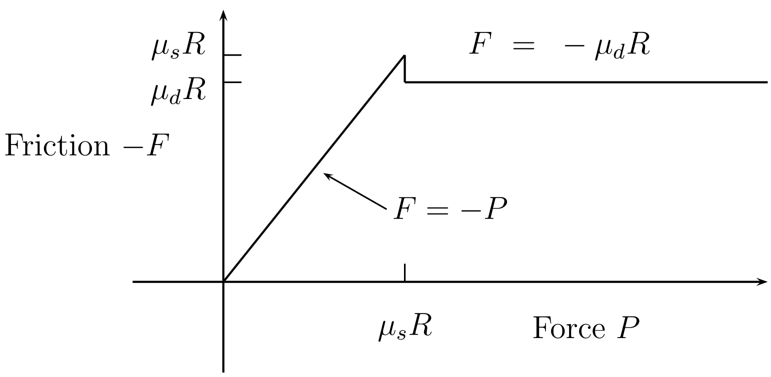
(iii) When the limiting value is reached friction cannot increase any further and motion is about to begin. (This is called 'limiting friction'.)

The value of the limiting friction is related to the normal reaction R by $F = \mu_s R$ where μ_s is the coefficient of static friction (a constant between 0 and 1, depending on the materials of the body and the surface).

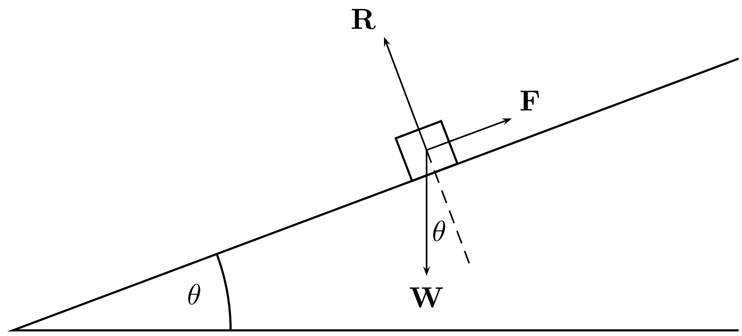
(iv) Once motion starts the friction is given by $F = \mu_d R$ where μ_d is the coefficient of dynamic friction, covered in the dynamics part of the course.

μ_d is usually slightly less than μ_s. Usually it will be obvious from the context which of μ_s and μ_d we are using and we shall just use μ.

These remarks can be summarised in the following diagram. Suppose we are applying a force P to a particle resting on a horizontal surface with normal reaction R, then the friction force (which is in the direction opposite to P) is given by



Example 1: A block of weight W is resting on a slope, inclined at angle θ to the horizontal. The coefficient of static friction is μ.



If we increase θ, at what angle will the block begin to slip?

Suppose that the block is in equilibrium. If we resolve parallel and perpendicular to the slope we get:

$$\text{Resolving } \perp \text{ to plane:} \quad R = W\cos\theta$$

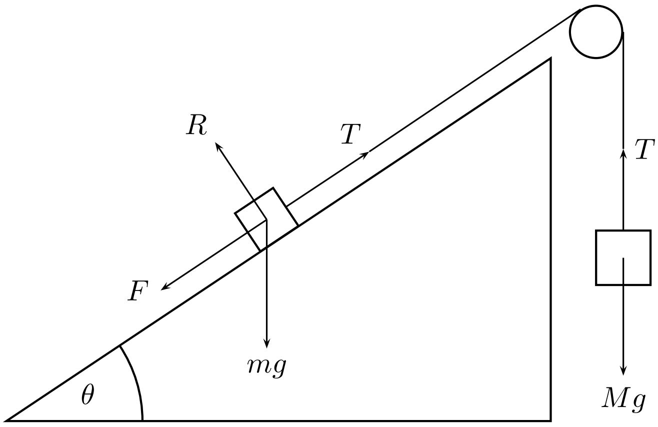
$$\text{Resolving } \parallel \text{ to plane:} \quad F = W\sin\theta = \frac{R}{\cos\theta}\sin\theta = R\tan\theta$$

Friction can keep the block in equilibrium so long as $F \leq \mu R$; that is, so long as

$$R\tan\theta \leq \mu R \qquad \text{or} \qquad \tan\theta \leq \mu.$$

The angle where $\tan\theta = \mu$ is the limiting case.

Example 2: A particle of mass m rests on a slope at angle θ, with coefficient of friction μ. It supports a mass M by a string passing over a fixed pulley as shown. What is the maximum M it can support?



(Note: The friction acts down the slope. This is because when M has its maximum value, the particle on the plane is just on the point of moving up the plane. The friction opposes this motion, so acts down the plane.)

For mass M, all the forces are vertical. For equilibrium we must have $T - Mg = 0$ so $T = Mg$.

For mass m, resolve parallel and perpendicular to slope:

Resolving $\perp$ to slope: $R - mg\cos\theta \;=\; 0$ so $R = mg\cos\theta$.

Resolving $\|$ to plane : $T - F - mg\sin\theta \;=\; 0$.

Since M is a maximum, motion is just about to occur, and we are in the limiting friction case. Hence $F = \mu R = \mu mg\cos\theta$. So

$$\begin{aligned} T - \mu mg\cos\theta - mg\sin\theta &= 0 \\ \Longrightarrow \quad Mg - \mu mg\cos\theta - mg\sin\theta &= 0 \\ \Longrightarrow \quad Mg &= \mu mg\cos\theta + mg\sin\theta \\ \Longrightarrow \quad M &= \mu m\cos\theta + m\sin\theta \end{aligned}$$

So the maximum M is $m(\mu\cos\theta + \sin\theta)$.